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Paley–Wiener Theorem for Singular Support of K-Finite Distributions on Symmetric Spaces

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We study Fourier transforms of distributions on a symmetric space X. Eguchi *et al.* [1] characterized the image of $\mathscr{E}'(X)$-distributions of compact support under the Fourier transform. We give a simpler proof of Eguchi's result and characterize the size of the singular support for the K-finite members of $\mathscr{E}'(X)$. We apply this Paley–Wiener type theorem to invariant differential equations on X.

INTRODUCTION

Let G be a noncompact semisimple Lie group with finite center and let $K \subset G$ be a maximal compact subgroup. Denote by $D(G/K)$ the set of invariant differential operators on the symmetric space $X = G/K$. S. Helgason showed in [3, 4] that the equation $Du = f$ with $D \in D(G/K)$ can be solved in the space $C^\infty(X)$ and the space of K-finite distributions (a distribution is K-finite if its left translates by $k \in K$ span a finite dimensional subspace). To study the solvability question in the space of all distributions one may attempt to employ Hörmander's criteria [3b]: The operator D is said to be strongly P-convex if for every compact set $C \subset X$ there is a compact set C', which may be taken empty if C is empty, so that if $T \in \mathscr{E}'(X)$ is a distribution of compact support, then

1. supp $DT \subset C \Rightarrow$ supp $T \subset C'$,
2. sing-supp $DT \subset C \Rightarrow$ sing-supp $T \subset C'$,

(where we recall that the singular support of a distribution is the complement of the set on which the distribution can be represented as a C^∞ function). Strong P-convexity then implies solvability. To characterize the size of the singular support of $T \in \mathscr{E}'(X)$ we employ the Fourier transform on X: Theorem 2.7 states that this size is related to the exponential type of the Fourier transform for K-finite distributions. This yields then a different proof of solvability in

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the space of K-finite distributions (Theorem 3.3). We give an example showing that the criteria in Theorem 2.7 (which are analogous to [5, Theorem 1.7.8] for Euclidean Fourier transform) do not suffice for general $T \in \mathscr{E}'(X)$.

It is possible to prove Theorem 2.7 by reducing it to the case when $T \in \mathscr{E}'(X)$ is K-invariant. This was kindly suggested to me by S. Helgason, who showed in [4] that the generalized spherical function is a certain derivative of the zonal spherical function ϕ_λ. We calculate these differential operators explicitly (Corollary 2.5) and use them to write down any K-finite distribution as a sum of derivatives of K-invariant ones (Lemma 2.6). In these calculations we rely heavily on results about K-finite functions in the representation space of spherical principal series of G (Konstant [7]) and their intertwining operators (Johnson and Wallach [11]).

Notation

Throughout this paper G will be a semisimple, noncompact connected Lie group with finite center. If $K \subset G$ is a maximal compact subgroup of G we write X for the symmetric space G/K, and $0 \in X$ for the coset eK. Let $\mathfrak{g}$ and $\mathfrak{k}$ be Lie algebras of G and K, respectively. We have then the Cartan decomposition of $\mathfrak{g}$, orthogonal with respect to the Killing form, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ and Σ (resp. Σ^+) corresponding restricted (resp. positive restricted) roots. The Killing form is nondegenerate on $\mathfrak{a}$. Let $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ be the subalgebras of $\mathfrak{g}$, $\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}^\alpha$, $\bar{\mathfrak{n}} = \theta(\mathfrak{n})$ where θ is the Cartan involution and $\mathfrak{g}^\alpha$ is the rootspace corresponding to α: $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X, \forall H \in \mathfrak{a}\}$. To Lie algebras $\mathfrak{a}$, $\mathfrak{n}$, $\bar{\mathfrak{n}}$ correspond simply connected closed subgroups of G denoted by A, N, $\bar{N}$, respectively. Let $\log\colon A \to \mathfrak{a}$ be the inverse of $\exp\colon \mathfrak{a} \to A$. We have the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and on the group level $G = KAN$. Every $g \in G$ decomposes uniquely $g = kan$ and we let $H(g) = \log a \in \mathfrak{a}$. Let M (resp. M') be the centralizer (resp. normalizer) of A in K and put $W = M'/M$. The Weyl group W is a finite group of linear automorphisms of $\mathfrak{a}$ and by duality of $\mathfrak{a}^*$ and $\mathfrak{a}_{\mathbb{C}}^*$—the complexification of $\mathfrak{a}^*$. Let $\rho \in \mathfrak{a}^*$ be $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$ where m_α is the multiplicity of the root α. We normalize the invariant measures on the various spaces as follows: For K and K/M we insist that their volumes be 1 and denote both measures by dk. On A (resp. $\mathfrak{a}^*$) we take $(\pi/2)^{-\ell/2}$-times (resp. $(1/|W|)(\pi/2)^{-\ell/2}$-times) the Euclidean measure induced by the Killing form, where $\ell = \dim A$ and $|W|$ is the order of the Weyl group. On $\bar{N}$ we choose $d\bar{n}$ so that $\int_{\bar{N}} e^{-2\rho H(\bar{n})}\, d\bar{n} = 1$ and we put $dn = \theta(d\bar{n})$ as the measure on N. Via the alternate Iwasawa decomposition $G = NAK$ the measures on K, A, N determine unique measures dg on G and dx on X so that the map $N \times A \to X$, $(n, a) \to na \cdot 0$ is a measure preserving diffeomorphism.

If $x \in X$ is written as $x = g \cdot 0$, $g \in G$ we put $A(x, kM) = -H(g^{-1}k) \in \mathfrak{a}$.

For every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the function $x \to \exp[(i\lambda + \rho)\, A(x, kM)]$ is an eigenfunction of all G-invariant (under left translations) differential operators on X. Denote the set of such operators by $D(G/K)$. In fact [2, pp. 92, 94] for $D \in D(G/K)$ we have

$$D_x e^{(i\lambda+\rho)A(x,kM)} = \Gamma(D)(i\lambda) e^{(i\lambda+\rho)A(x,kM)}$$

where $D \to \Gamma(D)$ is an algebra isomorphism of $D(G/K)$ onto the set of W-invariant polynomials on $\mathfrak{a}_{\mathbb{C}}^*$. By $\lambda \to \phi_\lambda(x)$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we denote the Harish–Chandra's parametrization of zonal spherical functions:

$$\phi_\lambda(x) = \int_K e^{(i\lambda+\rho)A(x,kM)}\, dk.$$

We frequently write $\mathscr{D}(X)$ for $C_c^\infty(X)$ and denote by $\mathscr{D}'(X)$ (resp. $\mathscr{E}'(X)$) the space of distributions (resp. distributions with compact support). For $g \in G$ and $f \in \mathscr{D}(X)$ we put $f^{\tau(g)}(x) = f(g^{-1}x)$ and if $T \in \mathscr{D}'(X)$ we let $T^{\tau(g)}(f) = T(f^{\tau(g^{-1})})$. For a differential operator D on X we define $D^{\tau(g)}$ by $(Df)^{\tau(g)} = D^{\tau(g)} f^{\tau(g)}$, $f \in C^\infty(X)$. Finally we call the set

$$B_R = \{x \in X \mid x = ka \cdot 0,\, k \in K,\, a \in A \text{ and } |\log a| \leqslant R\}$$

the ball of radius R.

1. Singular Support of K-Invariant Distributions

In this section we recall the Payley–Wiener theorems for $\mathscr{D}(X)$ and $\mathscr{E}'(X)$ and use them to characterize the size of singular support of K-invariant members of $\mathscr{E}'(X)$. The Fourier transform of $f \in \mathscr{D}(X)$ is defined (Helgason [5, p. 15]):

$$\tilde{f}(\lambda, kM) = \int_X e^{(-i\lambda+\rho)A(x,kM)} f(x)\, dx \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \qquad kM \in K/M$$

and is inverted by

$$f(x) = \int_{\mathfrak{a}^* \times K/M} e^{(i\lambda+\rho)A(x,kM)} \tilde{f}(\lambda, kM)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\, dk,$$

where $\mathbf{c}(\lambda)$ is the Harish–Chandra $\mathbf{c}$-function. For each $kM \in K/M$, $\lambda \to \tilde{f}(\lambda, kM)$ is a holomorphic function on $\mathfrak{a}_{\mathbb{C}}^*$ of exponential type:

Theorem 1.1 (Helgason [3]). *The Fourier transform is a linear isomorphism of $\mathscr{D}(X)$ onto the space of C^∞ functions $\psi(\lambda, kM)$ on $\mathfrak{a}_{\mathbb{C}}^* \times K/M$ holomorphic in λ and satisfying for some $R > 0$ and all $n \in \mathbb{Z}^+$:*

$$|\psi(\lambda, kM)| \leqslant c_n(1 + |\lambda|)^{-n} e^{R|\operatorname{im} \lambda|}, \qquad (\exists c_n > 0), \tag{1}$$

and such that $\lambda \to \int_{K/M} e^{(i\lambda+\rho)A(x,kM)}\psi(\lambda, kM)\,dk$ *is* W*-invariant* $\forall x \in X$. *Moreover the Fourier transform of* $f \in \mathscr{D}(X)$ *satisfies* (1) *iff* $\operatorname{supp} f \subset B_R$.

If $T \in \mathscr{E}'(X)$ its Fourier transform is defined $\tilde{T}(\lambda, kM) = T_x(e^{(-i\lambda+\rho)A(x,kM)})$ and again for each $kM \in K/M$, $\tilde{T}(\lambda, kM)$ is holomorphic and of exponential type on $\mathfrak{a}_{\mathbb{C}}^*$:

THEOREM 1.2 (Eguchi et al. [1]). *The Fourier transform is a linear isomorphism of* $\mathscr{E}'(X)$ *onto the space of* C^∞ *functions* $\Psi(\lambda, kM)$ *on* $\mathfrak{a}_{\mathbb{C}}^* \times K/M$, *holomorphic in* λ, *and satisfying for some* $R > 0$ *and* $N \in \mathbb{Z}$

$$|\Psi(\lambda, kM)| \leqslant c(1 + |\lambda|)^N e^{R|im\lambda|} \tag{2}$$

and

$$\int_{K/M} e^{(i\lambda+\rho)A(x,kM)}\Psi(\lambda, kM)\,dk = \int_{K/M} e^{(is\lambda+\rho)A(x,kM)}\Psi(s\lambda, kM)\,dk \tag{3}$$

for all $s \in W$ *and* $x \in X$. *Moreover the Fourier transform of* $T \in \mathscr{E}'(X)$ *satisfies* (2) *iff* $\operatorname{supp} T \subset B_R$.

The proof of this theorem in [1] depends on characterization of the Schwartz space on X, which has not appeared in print yet. We give a simpler proof based only on Theorem 1.1.

Proof of 1.2. Let $\eta_\epsilon(x) \in \mathscr{D}(X)$ be a spherical approximate identity (i.e., $\lim_{\epsilon\to 0} \eta_\epsilon = \delta_0$ delta function at the origin of X) so that $\operatorname{supp} \eta_\epsilon \subset B_\epsilon$. The Fourier transform of η_ϵ depends only on $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, and $\eta_\epsilon(\lambda) \to 1$ as $\epsilon \to 0$ uniformly on compact sets. For any $T \in \mathscr{E}'(X)$ we also have $(T * \eta_\epsilon)^\sim(\lambda, kM) = \tilde{T}(\lambda, kM)\,\tilde{\eta}_\epsilon(\lambda)$. Since $T * \eta_\epsilon \in \mathscr{D}(X)$ this and Theorem 1.1 show that the Fourier transform is injective on $\mathscr{E}'(X)$ (of course $T * \eta_\epsilon \to T$ in $\mathscr{E}'(X)$) and that $\tilde{T}(\lambda, kM)$ satisfies (3). To show that $\tilde{T}$ also satisfies (2) if $\operatorname{supp} T \subset B_R$, recall that Iwasawa decomposition implies that X is diffeomorphic to $N \times A$ and therefore to the Euclidean space $\mathfrak{n} \oplus \mathfrak{a}$ via the map $(X, H) \to \exp X \exp H \cdot 0$. Using these coordinates we can (by continuity of T) find differential operators D_i with constant coefficients on $\mathfrak{n} \oplus \mathfrak{a}$ so that $\forall \phi \in C^\infty(X)$

$$|T(\phi)| \leqslant \sup_{x\in B_R} \sum_{\text{finite}} |D_i\phi|(x).$$

Since $\tau(k)$ maps B_R into itself for any $k \in K$, we obtain

$$|T^{\tau(k)}(\phi)| \leqslant \sup_{x\in B_R} \sum |D_i^{\tau(k)}\phi|(x). \tag{4}$$

In $\mathfrak{n} \oplus \mathfrak{a}$ coordinates $D_i^{\tau(k)}$ are differential operators whose coefficients are real analytic on $K \times (\mathfrak{n} \oplus \mathfrak{a})$. Now in (4), taking $\phi = e^{(-i\lambda+\rho)A(x,eM)}$ and using the fact that differential operators "coming" from $\mathfrak{n}$ annihilate ϕ and for constant coefficient operators from $\mathfrak{a}$, ϕ is an eigenfunction with polynomial in λ as eigenvalue, we conclude that $\exists N \in \mathbb{Z}$ so that

$$|\tilde{T}(\lambda, kM)| = |T_x^{\tau(k^{-1})}(e^{(-i\lambda+\rho)A(x,eM)})| \leqslant c(1 + |\lambda|)^N e^{R|im\lambda|}.$$

Conversely, assume that $\Psi(\lambda, kM) \in C^\infty(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ is analytic in λ and satisfies (2) and (3). The Harish-Chandra $\mathbf{c}$-function is of polynomial growth on $\mathfrak{a}^*$ [3]. This fact together with (2) shows that the linear functional

$$\phi \to T(\phi) = \int_{\mathfrak{a}^* \times K/M} \Psi(\lambda, kM) \tilde{\phi}(-\lambda, kM) |\mathbf{c}(\lambda)|^{-2} \, d\lambda \, dk$$

is well defined on $\mathscr{D}(X)$. Let T_ϵ be the functional

$$T_\epsilon(\phi) = \int_{\mathfrak{a}^* \times K/M} \Psi(\lambda, kM) \, \tilde{\eta}_\epsilon(\lambda) \tilde{\phi}(-\lambda, kM) |\mathbf{c}(\lambda)|^{-2} \, d\lambda \, dk.$$

From (2) and definition of η_ϵ it follows that $\forall n \in \mathbb{Z}^+$, $\exists c_n > 0$ so that

$$|\Psi(\lambda, kM) \, \tilde{\eta}_\epsilon(\lambda)| \leqslant c_n (1 + |\lambda|)^{-n} e^{(R+\epsilon)|im\lambda|}. \tag{5}$$

By Theorem 1.1 $\Psi\tilde{\eta}_\epsilon$ is a Fourier transform of a test function, but the Plancherel formula [5]

$$\int_X f(x) \phi(x) \, dx = \int_{\mathfrak{a}^* \times K/M} \tilde{f}(\lambda, kM) \tilde{\phi}(-\lambda, kM) |\mathbf{c}(\lambda)|^{-2} \, d\lambda \, dk, \qquad \forall f, \phi \in \mathscr{D}(X),$$

implies that this test function considered as a distribution must be T_ϵ. By (5) we also have $\operatorname{supp} T_\epsilon \subset B_{R+\epsilon}$. But since $\lim_{\epsilon \to 0} T_\epsilon(\phi) = T(\phi)$ $\forall \phi \in \mathscr{D}(X)$, $T \in \mathscr{E}'(X)$ and $\operatorname{supp} T \subset B_R$.

Finally we check that $\tilde{T} = \Psi$:

$$\begin{aligned} \tilde{T}(\lambda, kM) &= T[e^{(-i\lambda+\rho)A(x,kM)}] = \lim_{\epsilon \to 0} T_\epsilon[e^{(-i\lambda+\rho)A(x,kM)}] \\ &= \lim_{\epsilon \to 0} \Psi(\lambda, kM) \, \tilde{\eta}_\epsilon(\lambda) = \Psi(\lambda, kM). \end{aligned}$$

Q.E.D.

We now turn to the singular support of distributions of compact support. Just like for the Fourier transform on Euclidean spaces (see Hörmander [35]) the above two theorems imply:

Proposition 1.3. *Let $T \in \mathscr{E}'(X)$ if* sing-supp $T \subset B_R$ *then there is an integer N and for each $m \in \mathbb{Z}^+$ a constant c_m so that*

$$|\tilde{T}(\lambda, kM)| \leqslant c_m (1 + |\lambda|)^N e^{R|im\lambda|} \qquad \text{if} \qquad |im\lambda| \leqslant m \log(1 + |\lambda|).$$

Proof. For each $m \in \mathbb{Z}^+$ wirte $T = S_m + \phi_m$ where $S_m \in \mathscr{E}'(X)$, supp $S_m \subset B_{R+1/m}$ and $\phi_m \in \mathscr{D}(X)$. Theorem 1.2 implies that for some integer N and a constant a_m

$$| \tilde{S}_m(\lambda, kM)| \leqslant a_m(1 + | \lambda |)^{N-1} e^{(R+1/m)|im\lambda|}. \tag{6}$$

Therefore if $| im\lambda | \leqslant m \log(1 + | \lambda |)$, (6) implies

$$| \tilde{S}_m(\lambda, kM)| \leqslant a_m(1 + | \lambda |)^{N} e^{R|im\lambda|}. \tag{7}$$

On the other hand by Theorem 1.1, $\tilde{\phi}_m(\lambda, kM)$ satisfies

$$| \tilde{\phi}_m(\lambda, kM)| \leqslant b_j^m(1 + | \lambda |)^{-j} e^{r|im\lambda|} \qquad \exists r > 0, \quad b_j^m \forall_j \in \mathbb{Z}^+.$$

Thus if $| im\lambda | \leqslant m \log(1 + | \lambda |)$ we get $| \phi_m(\lambda, kM)| \leqslant b_m^m$ which together with (7) proves the proposition.

We now consider the K-invariants in $\mathscr{E}'(X)$ which we denote by $\mathscr{E}_0'(X)$. If $T \in \mathscr{E}_0'(X)$ its Fourier transform $\tilde{T}(\lambda, kM) = \tilde{T}(\lambda)$ depends only on $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, and $\tilde{T}(\lambda)$ is W-invariant. For $T \in \mathscr{E}_0'(X)$ we have the converse to Proposition 1.3:

Proposition 1.4. *Let $\mathscr{E}_0'(X)$. Then* sing-supp $T \subset B_R$ *if and only if there is $N \in \mathbb{Z}$ and for each $m \in \mathbb{Z}^+$ a constant c_m so that*

$$| \tilde{T}(\lambda)| \leqslant c_m(1 + | \lambda |)^{N} e^{R|im\lambda|} \qquad \text{if} \qquad | im\lambda | \leqslant m \log(1 + | \lambda |). \tag{8}$$

Proof. If $\tilde{T}(\lambda)$ satisfies (8) we apply the Euclidean Paley–Wiener theorem for singular support (Hörmander, [5, Theorem 1.7.8]) to $\tilde{T}(\lambda)$ and conclude that on $\mathfrak{a}_{\mathbb{C}}^*$ $\tilde{T}(\lambda)$ is a sum of two holomorphic functions $\tilde{S}(\lambda)$ and $\tilde{\phi}(\lambda)$ of the following exponential types:

$$| \tilde{S}(\lambda)| \leqslant c(1 + | \lambda |)^{N} e^{R|im\lambda|}, \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*$$

$$| \tilde{\phi}(\lambda)| \leqslant c_n(1 + | \lambda |)^{-n} e^{r|im\lambda|}, \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*; \qquad \exists r > 0, \quad c_n \,\forall n \in \mathbb{Z}^+.$$

Now $\tilde{T}(\lambda)$ is W-invariant and since W is a finite group we can (by averaging over W) assume that $\tilde{S}(\lambda)$ and $\tilde{\phi}(\lambda)$ are also W-invaiiant. Theorems 1.1 and 1.2 then imply that $\tilde{\phi}(\lambda)$ and $\tilde{S}(\lambda)$ are Fourier transforms of $\phi \in \mathscr{D}(X)$ and $S \in \mathscr{E}_0'(X)$, respectively, with supp $S \subset B_R$. Q.E.D.

2. Other K-types

Let $\hat{K}_0$ be the set of unitary irreducible representations of K which when restricted to M contain a trivial representation. Fix a representation $\delta \in \hat{K}_0$. In this section we are going to characterize the size of singular support of distributions in $\mathscr{E}_\delta'(X)$, which is the subset of $\mathscr{E}'(X)$ whose elements transform under left translations by K according to δ. We reduce the problem to the strictly spherical case treated in the previous section. Let V_δ be a K-module on which K acts irreducibly according to δ. Let $d(\delta) = \dim V_\delta$ and $\ell(\delta) = \dim V_\delta{}^M$—the M-fixed vectors. Choose an orthonormal basis for V_δ, $v_1 \cdots v_{d(\delta)}$, so that the first $\ell(\delta)$ vectors form a basis for $V_\delta{}^M$. If $\check{\delta}$ denotes the contragradient representation choose $V_{\check{\delta}} = V_\delta^*$ (the dual vector space to V_δ) and take the basis of $V_{\check{\delta}}$ to be the dual basis $v_1^* \cdots v_{d(\delta)}^*$. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of the Lie algebra let $S(\mathfrak{p})$ be the symmetric algebra on $\mathfrak{p}$. The compact group K acts on $S(\mathfrak{p})$ by the adjoint action and under it $S(\mathfrak{p})$ decomposes as follows:

THEOREM 2.1 (Kostant–Rallis [8]). *$S(\mathfrak{p}) = I \otimes H$, I is the set of K-invariants in $S(\mathfrak{p})$ and $H = \sum_{\delta \in \hat{\mathfrak{k}}_0} H_\delta$ where each $u \in H_\delta$ transforms under K according to δ. The multiplicity of δ in H_δ is $\ell(\delta)$.*

Let $\sigma: S(\mathfrak{p}) \to \mathscr{U}(\mathfrak{g})$ be the symmetrization map. Write H^* for $\sigma(H)$ and H_δ^* for $\sigma(H_\delta)$. The universal enveloping algebra decomposes into $\mathscr{U}(\mathfrak{g}) = (\mathscr{U}(\mathfrak{g})\,\mathfrak{k} + \mathfrak{n}\mathscr{U}(\mathfrak{g})) \oplus \mathscr{U}(\mathfrak{a})$ (the second sum is direct). For $u \in \mathscr{U}(\mathfrak{g})$ let $q^u \in \mathscr{U}(\mathfrak{a})$ so that $u - q^u \in (\mathscr{U}(\mathfrak{g})\,\mathfrak{k} + \mathfrak{n}\mathscr{U}(\mathfrak{g}))$. Let $\epsilon_1 \cdots \epsilon_{\ell(\delta)}$ be a basis for $\mathrm{Hom}_K(V_\delta, H^*)$. We define $\ell(\delta) \times \ell(\delta)$ matrix of polynomial functions on $\mathfrak{a}_{\mathbf{C}}^*$

$$Q_{ij}^\delta(\lambda) = q^{\epsilon_j(v_i)}(\rho - i\lambda), \qquad \lambda \in \mathfrak{a}_{\mathbf{C}}^*.$$

In the basis $v_1 \cdots v_{\ell(\delta)}$, the matrix $Q^{\check{\delta}}(\lambda)$ is an element of $\mathrm{Hom}(V_\delta{}^M, V_\delta{}^M)$, which, however, depends on the choice of ϵ_j.

Let $\Lambda = \{\lambda \in \mathfrak{a}_{\mathbf{C}}^* \mid \mathrm{Re}\langle i\lambda, \alpha\rangle > 0,\ \forall \alpha \in \Sigma^+\}$. If $s \in W$ let $\bar{N}_s \subset \bar{N}$ be the subgroup $\bar{N}_s = m_s^{-1} N m_s \cap \bar{N}$ where $m_s \in M'$ is a representative of s. For $\lambda \in \Lambda$ and $s \in W$ let

$$A_s{}^\delta(\lambda) = \int_{\bar{N}_s} \delta(m_s(k(\bar{n}_s)))\, e^{(i\lambda-\rho)H(\bar{n}_s)}\, d\bar{n}_s \Big/ \int_{\bar{N}_s} e^{(i\lambda-\rho)H(\bar{n}_s)}\, d\bar{n}_s.$$

We have $A_s{}^\delta(\lambda) \in \mathrm{Hom}(V_\delta, V_\delta)$ and from the work of Helgason [2] and Schiffman [9] it follows that $A_s{}^\delta(\lambda)$ may be analytically continued from Λ to the whole of $\mathfrak{a}_{\mathbf{C}}^*$.

THEOREM 2.2 (Johnson and Wallach [11]). *For each $s \in W$ and $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ $A_s{}^\delta(\lambda)\, V_\delta{}^M \subset V_\delta{}^M$ and for a suitable choice of basis of* $\mathrm{Hom}_K(V_{\check{\delta}}, H^*)$

$$A_s{}^\delta(\lambda)\,|_{V_\delta^M} = Q^{\check{\delta}}(s\lambda)\, Q^{\check{\delta}}(\lambda)^{-1}.$$

Let $\mathscr{D}(X, \mathrm{Hom}(V_\delta, V_\delta))$ be C^∞ functions with compact support on X with values in $\mathrm{Hom}(V_\delta, V_\delta)$ and let $\mathscr{D}^\delta(X) = \{F \in \mathscr{D}(X, \mathrm{Hom}(V_\delta, V_\delta)) \mid F(kx) = \delta(k) F(x)\}$. Now K acts on $\mathscr{D}(X)$ by left translations. Denote by $\mathscr{D}_\delta(X)$ the K-finite functions of type δ under this action.

PROPOSITION 2.3 (Helgason [4]). *The mapping $f \to f^\delta$ where $f^\delta(x) =$*

$d(\delta)\int_K f(kx)\,\delta(k^{-1})\,dk$ is an isomorphism of $\mathscr{D}_{\check\delta}(X)$ onto $\mathscr{D}^{\delta}(X)$. The inverse is given by $f(x) = \mathrm{Tr}(f^{\delta}(x))$.

The Fourier transform of a function in $\mathscr{D}^{\delta}(X)$ is defined as

$$\tilde f^{\delta}(\lambda, kM) = \int_X e^{(-i\lambda+\rho)A(x,kM)} f^{\delta}(x)\,dx, \tag{1}$$

and can be expressed by means of the generalized spherical function of type δ:

$$\Phi_{\lambda,\delta}(x) = \int_K e^{(i\lambda+\rho)A(x,kM)}\delta(k)\,dk.$$

Indeed it is not too hard to check that letting $\tilde f^{\delta}(\lambda) = \tilde f^{\delta}(\lambda, eM)$ we obtain

$$\tilde f^{\delta}(\lambda) = \int_X \Phi^{*}_{\bar\lambda,\delta}(x)\, f^{\delta}(x)\,dx, \tag{1a}$$

where * means adjoint. One also verifies that $\tilde f^{\delta}(\lambda, kM) = \delta(k)\tilde f^{\delta}(\lambda)$ and that the range of $\tilde f^{\delta}(\lambda)$ is contained in $V_{\delta}{}^{M}$. The Fourier inversion for $\mathscr{D}^{\delta}(X)$ then becomes [3]:

$$f^{\delta}(x) = \int_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(x)\,\tilde f^{\delta}(\lambda)|\,\mathrm{c}(\lambda)|^{-2}\,d\lambda. \tag{2}$$

Helgason [4] has shown that as a consequence of irreducibility results, the matrix coefficients of the generalized spherical function $\Phi_{\lambda,\delta}$ can be expressed as derivatives of the zonal spherical function ϕ_λ : For $u, v \in V_{\delta}{}^{M}$ and almost all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ there is a right invariant differential operator L_λ depending on λ so that $\langle \Phi_{\lambda,\delta}(x)\,u, v\rangle = (L_\lambda \phi_\lambda)(x)$, $x \in X$. In the Corollary 2.5 we explicitly construct these differential operators. The dependence on λ turns out to be multiplication by rational functions in λ.

We define $I^{\delta}(X) = \{g \in \mathscr{D}(X), \mathrm{Hom}(V_{\delta}, V_{\check\delta}{}^{M})) \mid g(kx) = g(x)\, \forall k \in K\}$ and a $d(\delta) \times \ell(\delta)$ matrix of differential operators $D^{\check\delta}$:

$$D^{\check\delta} = \begin{pmatrix} \epsilon_1(v_1)^{\#} & \cdots & \epsilon_{\ell(\delta)}(v_1)^{\#} \\ \vdots & & \vdots \\ \epsilon_1(v_{d(\delta)})^{\#} & \cdots & \epsilon_{\ell(\delta)}(v_{d(\delta)})^{\#} \end{pmatrix}.$$

Since each $\epsilon_i(v_j)$ lies in $\mathscr{U}(\mathfrak{g})$ it is therefore naturally a right invariant differential operator on X. The sharp denotes the corresponding adjoint differential operator, with respect to the invariant measure on X. If $g \in I^{\check\delta}(X)$ let $g_{ij}(x) = \langle g(x)\,v_j^*, v_i^*\rangle$

be the matrix coefficients of g. $\check{D}^{\delta}$ then defines a map (by "matrix multiplication") from $\check{I}^{\delta}(X)$ to $\mathscr{D}(X, \mathrm{Hom}(V_\delta, V_\delta))$ by the formula:

$$(\check{D}^{\delta}g)_{ij} = \langle \check{D}^{\delta}gv_j^*, v_i^*\rangle = \sum_{\ell=1}^{\ell(\check{\delta})} \epsilon_\ell(v_i)^{\#} g_{\ell j}.$$

The ideas behind this lemma are due to Torasso [10].

LEMMA 2.4. *$\check{D}^{\delta}$ maps $\check{I}^{\delta}$ isomorphically onto $\check{\mathscr{D}}^{\delta}(X)$ and if $g \in \check{I}^{\delta}$ and $\tilde{g}(\lambda) = \int_X \phi_{-\lambda}(x)\, g(x)\, dx$ then $(\check{D}^{\delta}g)^{\sim}(\lambda) = Q^{\delta}(\lambda)\, \tilde{g}(\lambda)$.*

Proof. Since $g(x)$ is K-bi-invariant, easy calculation gives $(\check{D}^{\delta}g)(kx) = (\mathrm{Ad}(k^{-1})\, \check{D}^{\delta})\, g(x)$. But $\mathrm{Ad}(k^{-1})\, \epsilon_j(v_i)^{\#} = \epsilon_j(\delta(k^{-1})\, v_i)^{\#}$ for $\epsilon_j \in \mathrm{Hom}_K(V_\delta, H^*)$. Therefore if we denote the matrix entries of $\delta(k)$ by $\delta_{ij}(k) = \langle \delta(k)\, v_j, v_i\rangle$, we see that

$$(\mathrm{Ad}(k^{-1})\, \check{D}^{\delta})_{ij} = \epsilon_j(\delta(k^{-1})\, v_i)^{\#} = \sum_{k=1}^{d(\delta)} \delta_{ik}(k^{-1})\, \epsilon_j(v_k)^{\#} = \sum_{k=1}^{d(\delta)} \check{\delta}_{ik}(k)\, \epsilon_j(v_k)^{\#}$$

where $\check{\delta}_{ij}(k) = \langle \check{\delta}(k)\, v_j^*, v_i^*\rangle$. This together then proves that $\check{D}^{\delta}g$ lands in $\check{\mathscr{D}}^{\delta}(X)$:

$$(\check{D}^{\delta}g)(kx) = (\mathrm{Ad}(k^{-1})\, \check{D}^{\delta})\, g(x) = \check{\delta}(k)(\check{D}^{\delta}g)(x).$$

Next we show that $(\check{D}^{\delta}g)^{\sim}(\lambda) = Q^{\delta}(\lambda)\, \tilde{g}(\lambda) \in \mathrm{Hom}(V_\delta V_\delta{}^M) \quad \forall \lambda \in \mathfrak{a}_{\mathbf{C}}^*$.

$$(\check{D}^{\delta}g)_{ij}^{\sim}(\lambda) = \int_X e^{(-i\lambda+\rho)A(x,eM)} \left(\sum_{k=1}^{\ell(\delta)} \epsilon_k(v_i)^{\#}\, g_{kj}(x)\right) dx$$

$$= \sum_{k=1}^{\ell(\delta)} \int_X (\epsilon_k(v_i)\, \psi_\lambda)(x)\, g_{kj}(x) dx,$$

where

$$\psi_\lambda(x) = e^{(-i\lambda+\rho)A(x,eM)}$$

or

$$\psi_\lambda(na\cdot 0) = e^{(i\lambda-\rho)\log a}.$$

We decompose $\epsilon_k(v_i)$ according to $\mathscr{U}(\mathfrak{g}) = (\mathscr{U}(\mathfrak{g})\, \mathfrak{k} + \mathfrak{n}\mathscr{U}(\mathfrak{g})) \oplus \mathscr{U}(\mathfrak{a})$ and observe that since ψ_λ is annihilated by $\mathfrak{n}\mathscr{U}(\mathfrak{g})$ and since $g(x)$ is K-invariant $\int_X (\epsilon_k(v_i)\, \psi_\lambda)\, g_{kj} = \int_X (q^{\epsilon_k(v_i)}\psi_\lambda)\, g_{kj}$. But $q^{\epsilon_k(v_i)}\psi_\lambda = q^{\epsilon_k(v_i)}(\rho - i\lambda)\, \psi_\lambda = Q_{ik}^{\delta}(\lambda)\, \psi_\lambda$. Therefore

$$(\check{D}^{\delta}g)_{ij}^{\sim}(\lambda) = \sum_{k=1}^{\ell(\delta)} \int_X Q_{ik}^{\delta}(\lambda)\, \psi_\lambda(x)\, g_{kj}(x)\, dx = \sum Q_{ik}^{\delta}(\lambda)\, \tilde{g}_{kj}(\lambda).$$

It remains to prove that $\check{D}^{\delta}: \check{I}^{\delta} \to \check{\mathscr{D}}^{\delta}(X)$ is onto. Since we know ([2b], Lemma

6.3) that on V_δ for each $s \in W$, $\Phi_{s\lambda,\check{\delta}}(x) = \Phi_{\lambda,\check{\delta}}(x)\, A_s^{\check{\delta}}(\bar{\lambda})^*$ we have from (1a)

$$\tilde{f}^{\check{\delta}}(s\lambda) = A_s^{\check{\delta}}(\lambda)\, \tilde{f}^{\check{\delta}}(\lambda), \qquad s \in W. \tag{3}$$

Now (3) and Theorem 2.2 imply that $Q^\delta(\lambda)^{-1} \tilde{f}^{\check{\delta}}(\lambda)$ is W-invariant. But $\det Q^\delta(x)$ has no zeros (Kostant [7]) on a neighborhood of $(\mathfrak{a}^* + i\mathfrak{a}_+^*)$, where $\mathfrak{a}_+^* = \{\lambda \in \mathfrak{a}^* \mid \langle\lambda, \alpha\rangle > 0,\ \forall\alpha \in \Sigma^+\}$, and therefore, $\tilde{g}(\lambda) = Q^\delta(\lambda)^{-1} \tilde{f}^{\check{\delta}}(\lambda)$ is holomorphic on $\mathfrak{a}_{\mathbf{C}}^*$. Since Since $Q^\delta(\lambda)$ is a matrix of polynomials $\tilde{g}(\lambda)$ and $\tilde{f}^{\check{\delta}}(\lambda)$ are of the same exponential type (Hörmander [3a], Lemma 3.1.4). Therefore by Theorem 1.1

$$g(x) = \int_{\mathfrak{a}^*} \tilde{g}(\lambda)\, \phi_\lambda(x) \mid \mathbf{c}(\lambda)|^{-2}\, d\lambda$$

lies in $I^{\check{\delta}}$. By the first part of this lemma, however, $(D^{\check{\delta}}g)^\sim(\lambda) = Q^\delta(\lambda)\, \tilde{g}(\lambda) = \tilde{f}^{\check{\delta}}(\lambda)$. But the Fourier transform is injective and so $D^{\check{\delta}}g = f^{\check{\delta}}$, which concludes the proof. Q.E.D.

COROLLARY 2.5. *Restricted to $V_\delta{}^M$ the generalized spherical function equals*

$$\Phi_{\lambda,\delta}(x) = (D^\delta\phi_\lambda)(x)\, Q^{\check{\delta}}(\lambda)^{-1}, \qquad \lambda \in \mathfrak{a}^*.$$

Remark. If $x = a \cdot 0$, $a \in A$, then the image and the cokernel of $\Phi_{\lambda,\delta}(a)$ are $V_\delta{}^M$. Since any $x \in X$ is of the form $x = ka \cdot 0$, this corollary determines $\Phi_{\lambda,\delta}$ completely.

Proof. Let $g \in I^\delta$, then by Fourier inversion

$$g(x) = \int_{\mathfrak{a}^*} \phi_\lambda(x)\, \tilde{g}(\lambda) \mid \mathbf{c}(\lambda)|^{-2}\, d\lambda.$$

Let $f^\delta = D^\delta g$. Therefore,

$$f^\delta(x) = \int_{\mathfrak{a}^*} (D^\delta\phi_\lambda)(x)\, \tilde{g}(\lambda) \mid \mathbf{c}(\lambda)|^{-2}\, d\lambda. \tag{4}$$

From the above lemma we know that $\tilde{g}(\lambda) = Q^{\check{\delta}}(\lambda)^{-1} \tilde{f}^\delta(\lambda)$. By the inversion formula (2) we also have

$$f^\delta(x) = \int_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(x)\, Q^{\check{\delta}}(\lambda)\, \tilde{g}(\lambda) \mid \mathbf{c}(\lambda)|^{-2}\, d\lambda. \tag{5}$$

We claim that $\Phi_{\lambda,\delta}(x)\, Q^{\check{\delta}}(\lambda)$ is W-invariant on $\mathfrak{a}^*$: For we have $\Phi_{s\lambda,\delta} = \Phi_{\lambda,\delta} A_s{}^\delta(\bar{\lambda})^*$ and on $\mathfrak{a}^*$ the intertwining operator $A_s{}^\delta$ is unitary [3b] so for $\lambda \in \mathfrak{a}^*$, $\Phi_{s\lambda,\delta} = \Phi_{\lambda,\delta} A_s{}^\delta(\lambda)^{-1}$. By Theorem 2.2 we then have $\Phi_{s\lambda,\delta} Q^{\check{\delta}}(s\lambda) = \Phi_{\lambda,\delta} Q^{\check{\delta}}(\lambda)$ on $V_\delta{}^M$.

Thus if we let $S(\lambda, x) = [\Phi_{\lambda,\delta}(x)\check{Q}^{\delta}(\lambda) - (D^{\delta}\phi_{\lambda})(x)]$ we see from (4) and (5) that

$$\int_{\mathfrak{a}^*} S(\lambda, x)\,\tilde{g}(\lambda)\,\mathbf{c}(\lambda)^{-2}\,d\lambda \equiv 0, \qquad \forall g \in I^{\delta}(X), \quad x \in X.$$

Now $S(\lambda, x)$ is W-invariant and the matrix entries of $\tilde{g}(\lambda)$, $g \in I^{\delta}(X)$, are dense in the square integrable W-invariant functions on $\mathfrak{a}^*$ [2, Theorem 5.8]. This shows that $S(\lambda, x) \equiv 0$, and, since $\check{Q}^{\delta}(\lambda)$ is invertible on $\mathfrak{a}^*$, proves the lemma.

We now come to distribution of compact support and type δ. For $T \in \mathscr{E}'_{\check{\delta}}(X)$ we define an $d(\delta) \times d(\delta)$ matrix T^{δ} as follows:

$$T^{\delta}_{ij} = d(\delta) \int_K T^{\tau(k)}\,\delta_{ij}(k)\,dk.$$

We let $\mathscr{E}'^{\delta}(X)$ be the set of all $d(\delta) \times d(\delta)$ matrices M with entries in $\mathscr{E}(X)$ so that $M^{\tau(k)} = \sum_{\ell}^{d(\delta)} \delta_i\,(k)\,M_{\ell j}$. If $T \in \mathscr{E}'_{\check{\delta}}(X)$, then $T^{\delta} \in \mathscr{E}'^{\delta}(X)$ and $\mathrm{Tr}(T^{\delta}) = T$. The trace function $\mathrm{Tr}\colon \mathscr{E}'^{\delta}(X) \to \mathscr{E}'_{\check{\delta}}(X)$ is onto.

Next we define $I'^{\delta}(X)$ as the set of $\ell(\delta) \times d(\delta)$ matrices with coefficients in the K-invariant distributions of compact support. We define Fourier transforms of members of $\mathscr{E}'^{\delta}(X)$ and $I'^{\delta}(X)$ by taking Fourier transform of each matrix entry. It may be verified then that

$$\tilde{T}^{\delta}(\lambda) = \int_X \Phi^*_{\bar{\lambda},\delta}\,dT, \qquad \forall T \in \mathscr{E}'_{\check{\delta}}(X)$$

The following lemma is analogous to 2.4, although it *cannot* be obtained from it by duality.

LEMMA 2.6. *$D^{\delta}\colon I'^{\delta}(X) \to \mathscr{E}'^{\delta}(X)$ is an isomorphism onto and $(D^{\delta}S)^{\sim}(\lambda) = \check{Q}^{\delta}(\lambda)\,\tilde{S}(\lambda)$, $\forall S \in I'^{\delta}(X)$.*

Proof. The facts that D^{δ} maps $I'^{\delta}(X)$ into $\mathscr{E}'^{\delta}(X)$ and that $(D^{\delta}S)^{\sim}(\lambda) = \check{Q}^{\delta}(\lambda)\,\tilde{S}(\lambda)$ follow directly from Lemma 2.4 by convolving with a spherical approximate identity.

To prove that D^{δ} is onto, let $T^{\delta} \in \mathscr{E}'^{\delta}(X)$. Again from the fact that $\Phi_{s\lambda,\delta} = \Phi_{\lambda,\delta}A_s(\bar{\lambda})^*$ and Theorem 2.2 we get that $\tilde{S}(\lambda) = \check{Q}^{\delta}(\lambda)^{-1}\,\tilde{T}(\lambda)$ is W-invariant. By Theorem 1.2, $\tilde{S}(\lambda)$ is then a Fourier transform of $S \in I'^{\delta}(X)$. (We again used the fact that multiplication by polynomials does not alter the exponential type.) Finally we have that $(D^{\delta}S)^{\sim}(\lambda) = \check{Q}^{\delta}(\lambda)\,\tilde{S}(\lambda) = \tilde{T}^{\delta}(\lambda)$, proving that $D^{\delta}S = T^{\delta}$.

For K-finite distributions of $\mathscr{E}'(X)$ we now prove the converse to the Proposition 1.3:

THEOREM 2.7. *Let $T \in \mathscr{E}'_{\check{\delta}}(X)$. Then singular support of T is contained in a* *ball of radius R iff $\tilde{T}(\lambda, kM)$ satisfies the following estimate: For some integer N and constants c_m, $m \in \mathbb{Z}^+$*

$$|\tilde{T}(\lambda, kM)| \leqslant c_m(1 + |\lambda|)^N e^{R|im\lambda|} \qquad \text{if} \qquad |im\lambda| \leqslant m \log(1 + |\lambda|). \quad (6)$$

Proof. Since $\tilde{T}^\delta(\lambda)_{ij} = d(\delta) \int_K \tilde{T}(\lambda, kM)\, \delta_{ij}(k^{-1})\, dk$ it follows that $\tilde{T}^\delta(\lambda)_{ij}$ also satisfies (6). Again, since multiplication by polynomials does not change the esponential type of the estimate (6), we have that the matrix entries of $\tilde{S}(\lambda) = \tilde{Q}^\delta(\lambda)^{-1}\, \tilde{T}^\delta(\lambda)$ obey (6). Lemma 2.6 and Proposition 1.4 then insure that $S \in I'^\delta(X)$ and that matrix entries of S have singular support contained in B_R. But now also by Lemma 2.6 $T = \operatorname{Tr}(D^\delta S)$. Because differential operator D^δ cannot increase the size of singular support, the theorem follows.

From Theorem 2.7 it follows that if $T \in \mathscr{E}'(X)$ and T is K-finite, then we can characterize the size of singular support of T by the exponential growth of $\tilde{T}(\lambda, kM)$ on $\mathfrak{a}_{\mathbb{C}}^*$. If we drop the assumption on K-finiteness, such simple characterization is no longer possible. We give an example where T is not K-finite and $\tilde{T}(\lambda, kM)$ satisfies (6), but sing supp $T \not\subset B_R$: Let $G = S\ell(2, \mathbb{R})$. In its KAN decomposition we parametrize $A = \{a_t \mid a_t = \exp tH,\ H = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}\}$ and $K = \{k_\theta \mid k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\}$. If $\alpha \in \mathfrak{a}^*$ is the restricted root we identify $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $z \in \mathbb{C}$ by $\lambda = z\alpha$. On A and $\mathfrak{a}_{\mathbb{C}}^*$ we put a metric induced by *twice* the Killing form, which has the advantage that $|a_t| = |t|$ and $|\lambda| = |z|$. We calculate

$$e^{-(i\lambda+\rho)A(a_t, k_\theta M)} = e^{(iz-1/2)\ln(e^{-t}\cos^2\theta + e^t \sin^2\theta)}.$$

Now we take a function $f(t) \in C_c^\infty(\mathbb{R})$, $\operatorname{supp} f = [-1/2, 1/2]$, $f \geqslant 0$ and $f(t) = f(-t)$. We define $T_f \in \mathscr{E}'(X)$ by

$$T_f(\phi) = \int_{\mathbb{R}} f(t)\, \phi(a_t \cdot 0)\, dt, \qquad \phi \in \mathscr{D}(X).$$

Therefore, sing supp $T_f = \{a_t \cdot 0 \mid |t| \leqslant 1/2\}$. The Fourier transform of T_f is

$$\tilde{T}_f(z, \theta) = \int f(t)\, e^{(iz-1/2)\ln(e^{-t}\cos^2\theta + e^t\sin^2\theta)}\, dt. \quad (7)$$

Due to symmetry it is enough to know $\tilde{T}_f(z, \theta)$ for $\theta \in [0, \pi/4]$. If $\theta \in [\pi/6, \pi/4]$ and $t \in [-1/2, 1/2]$ then $|e^{-t}\cos^2\theta + e^t \sin^2\theta| \leqslant e^R$, $R < 1/2$. Therefore for $\theta \in [\pi/6, \pi/4]$, we have

$$|T_f(z, \theta)| \leqslant c e^{R|imz|}. \quad (8)$$

Next, if $\theta \in [0, \pi/6]$, $t \to \ln(e^{-t}\cos^2\theta + e^t\sin^2\theta)$ has nonvanishing derivative on a neighborhood of $[-1/2, 1/2]$. We set $u(t, \theta) = \ln[e^{-t}\cos^2\theta + e^t \sin^2\theta]$, solve for t and (7) becomes:

$$\tilde{T}_f(z, \theta) = \int f(t(\theta, u))\, e^{(iz-1/2)u} \frac{\partial t(\theta, u)}{\partial u}\, du,$$

or

$$\tilde{T}_f(z, \theta) = \int F(\theta, u)\, e^{izu}\, du \qquad \text{where} \quad F(\theta, u) = f(t(\theta, u))\, e^{-u/2} \frac{\partial t(\theta, u)}{\partial u}.$$

Since $f \in C_c^\infty(\mathbb{R})$ with support in $[-1/2, 1/2]$, $F(\theta, u)$ extends to a C_c^∞ function on $[0, \pi/6] \times \mathbb{R}$. It follows then from classical Paley–Wiener theorem [5, Theorem 1.7.7] that $T(z, \theta)$ is bounded on each set $\{(\theta, z) \mid \theta \in [0, \pi/6] \mid |imz| \leqslant m \log(1 + |z|)\}$. This combined with (8) gives that $\forall m \in Z^+$, $\exists c_m$

$$|T_f(z, \theta)| \leqslant c_m e^{R|imz|} \qquad \text{if} \qquad |imz| \leqslant m \log(1 + |z|).$$

Since $R < 1/2$ but sing supp T_f extends to the boundary of the ball of radius $1/2$ we have the desired counterexample.

3. Application to Differential Equations

Let $D \in D(G/K)$, the set of G-invariant differential operators on X. In [3, 4] Helgason has shown that both $D\colon C^\infty(X) \to C^\infty(X)$ and $D\colon \mathscr{D}_\delta'(X) \to \mathscr{D}_\delta'(X)$, $\delta \in \hat{K}_0$ are onto. In this section, we are going to give a different proof of the second result based on the characterization of singular supports established in Section 2. This is an application of standard techniques of functional analysis. All we have to show is that every $D \in D(G/K)$ does not "decrease" the support and singular support of members in $\mathscr{E}_\delta'(X)$. More precisely, we have:

Theorem 3.1. [3]. *Let $T \in \mathscr{E}'(X)$ and $D \in D(G/K)$. Then* supp $DT \subset B_R$ *implies* supp $T \subset B_R$.

Theorem 3.2. *Let $T \in \mathscr{E}_\delta'(X)$ and $D \in D(G/K)$. If* sing supp $DT \subset B_R$ *(resp. is empty) then* sing supp $T \subset B_R$ *(resp. is empty).*

Proof. Let $S = DT$. On the Fourier transform side we have $\tilde{S}(\lambda, kM) = \Gamma(D)(i\lambda)\, \tilde{T}(\lambda, kM)$ where $\Gamma(D)$ is a W-invariant polynomial on $\mathfrak{a}_{\mathbb{C}}^*$. $\tilde{S}$ and $\tilde{T}$ are, therefore, of the same exponential type on $\mathfrak{a}_{\mathbb{C}}^*$ and Theorem 2.7 then states that sing supp T and sing supp S are contained in a ball of the same radius.

In the case when $S \in C_c^\infty(X)$, $\tilde{S}(\lambda, kM)$ satisfies (Theorem 1.1) $\forall n \in \mathbb{Z}^+$ and some $r > 0$.

$$|\tilde{S}(\lambda, kM)| \leqslant c_n(1 + |\lambda|)^{-n} e^{r|im\lambda|}. \tag{1}$$

$\tilde{T}(\lambda, kM)$ then also satisfies (1) with possibly different constants and, therefore, (by Theorem 1.1) T is actually infinitely differentiable function.

Theorems 3.1 and 3.2 state that every invariant differential operator $D \in D(G/K)$ is strongly P-convex on the space $\mathscr{D}'_\delta(X)$ (for definition see Introduction and Hörmander [6]). This implies by [6, Theorem 1.2.4] the following solvability theorem (see also [4, Theorem 8.1]).

THEOREM 3.3. *Let $D \in D(G/K)$. Then for every $\delta \in \hat{K}_0$, $D: \mathscr{D}'_\delta(X) \to \mathscr{D}'_\delta(X)$ is onto.*

Restated Theorem 3.3 says that the equation $DT = S$ can be solved for every $D \in D(G/K)$ and S a K-finite distribution on the symmetric space X.

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